Painlevé in 1921, a breaking-through solution, in general relativity, totally misunderstood at that time !

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1 SUMMARY

Paul Painlevé (1863-1933), known as a politician, is also a mathematician. Initially circumspect about the theory of relativity, he published three notes at the Academy of Sciences. In the first one, in October 24, 1921 [11] he explains the method that he will adopt for comparing classical mechanics and general relativity and he announces some results that he will explicit in the second note November, 14 , 1921[12]. The last one, written on May 1 1922 [13] after the visit of Einstein in Paris, is a synthesis of his contribution. In [12], at first, he proposed a metric unifying Newtonian mechanics equations of geodesic motion in only one equation, like in the space-time equation of general relativity.

Then, focusing his analysis to the mathematical formalism of the general relativity, he develops an infinite class of relativistic solutions, in spherical coordinates, to the Schwarzschild's problem. This allows to describe the solution of Schwarzschild. This allows also to describe other solutions, such as his own proposal in [11], describing the same spacetime that Schwarzschild, but without singularity on the horizon, and also a symmetrical spacetime of that of Painlevé's (obtained by inverting the sign of the $dt.dr$ term). Together, Painlevé's spacetime and its symmetrical spacetime describe the full spacetime in general relativity for this problem (four regions). In the third note he will extend to general relativity the geometric formalism that he used in Newtonian theory for representing the Schwarzchild's geodesic equation by an equation in space multiplied by a conformal factor.

Then, we will review the debate between Einstein and his colleagues in Paris in April 1922 and we will comment the debate at the Académie of Sciences. In the conclusion we will wonder how Painlevé, who was reluctant to the relativity, was able to propose an innovative contribution and why his contemporaries have been unable to understand it.

2 INTRODUCTION

The main history of general relativity does not exhibit contributions of french scientists. Does this mean that they all missed the opportunity to contribute to this monument of the modern science ? As the contribution of Painlevé, among some of his colleagues of the Academy of Sciences, testifies , they did contribute but likely their contributions, whose interest will be acknowledged too far later, were too innovative for being understood at that time.

3 GENERAL RELATIVITY AND THE ACADEMY OF SCIENCES

Einstein publishes his definitive equations in November 1915, in the midst of the world conflict, while he is a professor in Berlin. Needless to say that under these conditions, the publication of Einstein did not get the attention of the Academy of Sciences which, in Paris, is mobilized for the war effort. The situation will change in 1921 as, on the one hand because Herman Weyl, a renowned mathematician, devotes part of his book "Zeit-Raum-Materie" to general relativity, which is translated into French in 1922 under the title " Espace-Temps-Matière "[1] and secondly because Einstein is rewarded by the Nobel Prize, even though it is not for general relativity. At first, the general relativity, considered as destroying the classical mechanics by the Academy of Sciences, is not welcome (Le Roux [8], [9]).

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This will however evolve over time. Paul Langevin [10], convinced that the time of reconciliation had come, at least between scientists, is the first in November 1921 ([4]) to defend the theory of general relativity. A lively debate follows, with 12 contributions on general relativity in 1921, 19 in 1922 and 9 in 1923. Langevin, competent and efficient ambassador, will continue to gradually create a trend favorable to the ideas of Einstein within the Academy. In this context, Paul Painlevé proposes, in a first report to the Academy of Sciences on October 24, 1921 ([5]), to compare the two theories, that of Newton and that of Einstein. This first article, rather critical, but constructive is an announcement of his work which he will explicit in a second article, shortly after, on November 14, 1921 ([6]). He will write a third one on May 1922 ([7]), after the debate with Einstein and his colleagues, at the Collège de France, during Einstein's visit to Paris (March 30-April 7, 1922). Recalled by his political career, he will withdraw from the debate after May 1, 1922. His contribution only lasted 6 months.

Figure 1: Painlevé in 1923: Known as a politician, since he held various important ministerial positions during the third republic, including head of the government first in 1917 and secondly in 1925 and once President of the Parliament (1924-1925). He died in October 1933. After a national funeral, he was buried in the Pantheon. Student of the Ecole Normale Supérieure, graduated in mathematics in 1886, he then studied mathematics with Hermann Schwarz and Felix Klein in Göttingen and returned to teach in France as a professor (University of Paris, Polytechnique, Collège de France, ENS). His brilliant work on differential equations was acknowledged by his election at the Academy of Sciences in 1900 (at age 37). Photo Press agency Meurisse, BNF collection

4 A GEOMETRICAL FORMALISM OF NEWTONIAN GRAVITATION FROM WHICH EMERGES A PHYSICAL TIME

For the problem of the gravitational field outside a single material body with spherical symmetry, in order to allow an efficient comparison of Newtonian gravitation and relativistic gravitation, as the latter is a geometric theory of gravitation, Painlevé proposes a formalism, also geometric but strictly spatial for the Newtonian gravitation. In his article of November 1921, [12], p.876, he states[1]:

"'It follows from this, as we see, that we can give the theory of Newtonian gravitation the following form (principle of the least action): The trajectories of the point P are the geodesics of the ds^2"'

$$ds^2 = (U + h)(dx^2 + dy^2 + dz^2) \tag{1}$$

"'where h is an arbitrary constant, U which is the Newtonian potential is a function of x, y, z which vanishes at infinity whose ΔU is zero outside the sphere S and is equal to a negative constant in S."'

We note that this equation (1), purely spatial, is the product of the Euclidean ds^2 by the conformal factor $(U + h)$.

[1]Translated Painlevé's articles extracts are in italic

CONFORMAL FACTOR

It is interesting to compare non-identical space-times, but which have, for example, in common an important phenomenology such as the causal structure. The metrics of these spacetimes will be linked by a conformal factor which is simply a function of the coordinates. Thus, if ds^2 is one of them, the other dS^2 will be of the form $dS^2 = f^2(x^\mu)ds^2$ where f is a function and where x^μ denotes the coordinates used to describe the two spacetimes. This factor plays here the role of a gauge in the sense that H. Weyl [7] defines it, at that time. One usually define the spatial geometry of the (plane) trajectory in polar coordinates by the equation $r(\varphi)$. But we can also define it parametrically $r(\lambda)$ and $\varphi(\lambda)$ where λ is the affine parameter of this trajectory. Since the equation $r(\varphi)$ fully define the spatial curve, the parameter λ is free and Painlevé can apply $(U + h)$ on the affine parameter usually defined in Euclidean metric, as a gauge, for generating a parameter λ proportional to the real time τ of a physical body traveling on this trajectory.

4.1 THREE REMARKABLE PROPERTIES FOLLOW

1. The formalism is unified: A single equation describes the geodesic instead of two. In Newtonian formalism, where time is universal and independent of space, two equations are necessary to describe the geodesic motion: the first one describes spatial geometry (a curve in a plane which can be an ellipse, a parabola or an hyperbola, that can be degenerated), the affine parameter of which is Euclidean, and the second one describes the geodesic motion on this spatial curve (law of the areas). A single equation is only necessary in the geometric formalism developed by Painlevé.

2. We can define a proper time which is a time no longer universal and absolute but dedicated only to the geodesic, like in relativity, proportional (equal, with an adequate parametrization) to the dynamic parameter of the spatial geodesic, which is a spacelike parameter. This contribution is remarkable. This (proper) time, which does not refer to Newtonian universal time, emerges from physics, since it is the conformal factor $(U+h)$, acting as a gauge on the affine parameter, where U is the gravitational potential and h is the conserved energy on the geodesic, which determines it. This affine parameter, which is the observer's proper time, on the geodesic determined by gravitation, is its dynamic parameter. This is a major advance in understanding the nature of time in physics. The philosophical debate of the primacy of the concepts among time, space and motion, which are linked by a relation, finds here a solution similar to that proposed by relativity.

3. This proper time appears as an "imaginary" parameter in the equations, (with a factor i) with respect to the space, like in the relativistic form, where the signature of time in the ds^2 is opposite to that of space. This shows that when we want to unify time and space in equations, the difference between these two concepts is formally exhibited in the equations by this property, as in relativity.

5 GENERALIZATION OF THIS FORMALISM TO GENERAL RELATIVITY

Painlevé will propose this generalization in his note [13], written after the debate with Einstein in Paris where he had to face a total disagreement of the scientific communality for the oriented solution of the metric form, that he proposed in his first note [11]. Therefore, Painlevé has serious doubts on the validity of his proposal. But as a mathematician, he his sure that the formal derivation of his contested metric is correct, so the problem should be physical: The nature should not permit such solution. This is why he will invoke (unduly) a "principle of reversibility". This shows how the general relativity and its implications were not well understood at that time. Thus, Painlevé will propose a solution, which will be a generalization to general relativity of the method he had used successfully for the Newtonian problem, with only quadratic terms invoking (unduly) the reversibility of time in postulate VI, [13], page 1141.

Painlevé statement, in modern notations: *POSTULAT VI: The ds^2 will not change when we change t to -t (reversibility principle). The ds^2, with spatial indexes in latin letters (varying from 1 to 3), is then necessarily of the form:*

$$ds^2 = \frac{dt^2}{V(x^i)} - d\sigma^2 \tag{2}$$

Where $d\sigma^2 = g_{jk}dx^j.dx^k$ as in the Euclidean case, but is no longer Euclidean.

Comparing that with the Schwarzschild's metric, and per the usual definition of U in general relativity, this induces to set:

$$V = \frac{1}{1 - \frac{2GM}{r}} = \frac{1}{U} \tag{3}$$

Let us notice that Painlevé uses $V(x^i) = U(x^i)^{-1}$, where $U = (1 - 2GM/r)$ in polar coordinates is the usual notation for the potential in the Schwarzschild's form, in relativity.

We know that per the correlation between the Hamilton principle and the least action principle that the trajectories of the point P are then given by the geodesics of the ds_1^2 of three spatial coordinates.

$$ds_1^2 = (U + h)(d\sigma)^2 \tag{4}$$

where h is an arbitrary constant and t is defined by:

$$dt = \frac{V.d\sigma}{\sqrt{V + h}} \Leftrightarrow dt^2 = \frac{V^2.d\sigma^2}{V + h} = \frac{d\sigma^2}{(U^2)(\frac{1}{U} + h)} = \frac{d\sigma^2}{(U)(1 + h.U)} \tag{5}$$

In his articles, it is not clear where Painlevé derived the above equation, but it is correct.[2]

Inserting the value of dt^2 in equation (2), with $U = 1/V = 1 - 2GM/r$, as stated by equation (3), and by setting $h = -1/E^2$ yields:

$$ds^2 = U\frac{d\sigma^2}{(U)(1 + h.U)} - d\sigma^2 = d\sigma^2(\frac{1}{1 + h.U} - 1) = d\sigma^2\frac{-h.U}{1 + h.U} = d\sigma^2\frac{-U}{\frac{1}{h} + U} = d\sigma^2\frac{-U}{-E^2 + U} = d\sigma^2\frac{U}{E^2 - U} \tag{6}$$

The trajectories of light are obtained by setting h to zero, therefore $dt = \sqrt{U}.d\sigma$.This implies $ds^2 = 0$

Physical parameters in these equations: In these equations, where $U = 1 - 2GM/r$ is the relativistic potential and E is the relativistic energy, conserved on a geodesic, we will get the same geodesic equation than that derived from the Schwarzschild form of the metric. In this geodesic equation we will include the angular momentum $L = r^2 d\varphi/ds$ also conserved on the geodesic. Energy E and angular momentum L are conserved on the geodesic as the metric does not depend on the coordinates t and φ. For the geodesic equation, as the space section of the geodesic is included in a plane, per the spherical symmetry, we can set $\theta = \pi/2$. In relativistic equations, for simplifying the notation, as usual we will set $c = 1$. Let us notice that we established this equation without using the equation (4) which looks unnecessary.

$$ds^2 = (\frac{U}{E^2 - U})(d\sigma)^2 = (\frac{1 - \frac{2GM}{r}}{E^2 - (1 - \frac{2GM}{r})})(\frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\varphi^2) \tag{7}$$

Dividing by ds^2, inserting the angular momentum L and simplifying the equation (7) yields:

$$E^2 - (1 - \frac{2GM}{r}) = \frac{dr^2}{ds^2} + (1 - \frac{2GM}{r})\frac{L^2}{r^2} \Rightarrow -E^2 + \frac{dr^2}{ds^2} + (1 - \frac{2GM}{r})(\frac{L^2}{r^2} + 1) = 0 \tag{8}$$

This is the geodesic equation in the Schwarzschild solution. This form is interesting as it generalizes the method used for getting the geometric form of the Newtonian mechanics. The relativistic spacetime geodesic is derived from the space metric, which is no longer Euclidean in this case, multiplied by a conformal factor. This is a non trivial property of this solution.

6 THE RELATIVISTIC FORM OF PAINLEVÉ

In its first article, Painlevé proposes a new form of metric, alternative to that proposed by Schwarzschild, he wrote [11] p.679:

[2]This can be verified by using the definition of the energy of a unitary mass in the Schwarzschild's spacetime by the equation $E = Udt/ds \rightarrow E^2 = U^2 dt^2/ds^2$. and identifying this ds^2 to that of the equation (2).

"'With this hypothesis, the Einsteinians propose the, now famous, ds^2 (four variables) whose geodesics define in their theory the motion of a massive point which reads :"'

$$ds^2 = (1 - \frac{a}{r})dt^2 - (\frac{1}{1 - \frac{a}{r}})dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (9)$$

"'But this ds^2 is not the only one that meets all the Einsteinian conditions. There is an infinity of others depending on two functions of r, and the choice of the formula (9) between all these formulas is purely arbitrary. These formulas are as simple as formula (9) and involve exactly the same verifications. For instance, we can use instead"':

$$ds^2 = (1 - \frac{a}{r})dt^2 - 2\sqrt{\frac{a}{r}}dr.dt - [dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \qquad (10)$$

"'where a denotes an arbitrary constant that depends on the mass of the material center O."'

A non singular solution on the horizon: Painlevé propose the first form of metrics for this problem, not singular on the horizon at $r = 2GM/c^2$. This should have pleased the relativists, which tried to elude it by considering it as a mathematical artefact, but they did not understand it, relying rather on the impossibility for the nature to engender of such objects, following Eddington who declared *that the nature could not allow such a monstrosity.* We corrected a misprint in the equation listed in his article [11] where we read the sign "'+"' for the cross term $dr.dt$. This is the sign "'-"' which is requested for describing the same spacetime than the Schwarzschild's spacetime.

Painlevé derived in, 1921, the maximally analytic solution: As we will see, in more details, in chapter 7, when considering the description by Painlevé of the method he has used for deriving this equation, the solution with the sign "'+"' is also a solution, but which describes the symmetrical counterpart of the Schwarzschild spacetime in a "'maximally analytic solution"'covering the whole spacetime. Far before Kruskal which in the 60's [30] will give an unified description of it, the Painlevé form includes the description of it by the means of two equations differing only by the sign of the cross term $dr.dt$.

It should be noted that a few months later Gullstrand in 1922 [20] also proposed a similar solution.

Figure 2: Karl Schwarzshild (1873-1916), Famous astronomer in Postdam (Germany). Lieutenant of artillery on the Russian front, he had the opportunity to read the final development of the theory of general relativity, in Einstein's articles published in November 1915, in the Preussische Akademie der Wissenschaften Sitzungberichte. Einstein presents the results of Schwarzschild at the Academy of Sciences of Prussia on January 13, 1916. A few months later in June, Schwarzschild dies in Potsdam as a result of a disease contracted at the front.

SINGULARITY OF SCHWARZSCHILD

General relativity is a geometric (non-Euclidean) four-dimensional (three of space and one of time) theory of gravitation. In this theory, the bodies travel on the geodesics (curves) of this geometry. The calculation of the parameters of these geodesic curves, for example the travel time of an observer between two points of a trajectory, requires a metric tensor denoted in general ds^2. For the calculations it is convenient to use the tools of the analytic geometry. This implies to define this metric tensor in some (arbitrary) coordinates. Even though, the geodesics do not depend on the coordinates used, some coordinates may reveal more clearly the structure of the spacetime. Equation (9) describes what we call the "Schwarzschild" [6] metric because he was the first in history to propose a solution for this problem, even though equation (9) is an extension, provided a few months later by Droste [26] of Schwarzschild's original metric, as it is described in [21]. This metric provides the solution of spacetime generated by a single body with spherical symmetry, outside of this body, in some coordinates. This metric is fully convenient for describing the solar system.
Original Schwarzschild solution is restricted to the exterior of the horizon, as Schwarzschild was relying on preliminary articles of Einstein in 1915,[1], [2], [3],[4], constraining the determinant of the metric to be equal to -1, this discarding the use of spherical coordinates, preferred choice for this problem exhibiting a spherical symmetry of space, Schwarzschild built some hybrid polar coordinates which unfortunately did not cover the whole spacetime. In equation (9), the second term of the right-hand side becomes infinite if $a = r$. Since no physical quantity can be infinite, this is what is called a singularity.
The form of metric (10), describing the same space-time, but in other coordinates, proposed by Painlevé, which is oriented (conceptual character), does not have this defect: This shows that this singularity is not physical and is just implied by an unduly constraint (the non-orientation of time) that has been unduly imposed. Lemaitre in chapter 11 of [27] will provide a very convincing analysis of this problem.

7 THE METHOD USED BY PAINLEVÉ FOR BUILDING ITS METRIC FORM AND ITS RESULT

7.1 Painlevé explains his method

As a mathematician, Painlevé understood how to build an equation consistent with general relativity. He thus starts from a stationary generic form with spherical spatial symmetry, depending therefore only on coordinate r, the ds^2 of which must be of the form:

$$ds^2 = A(r)dt^2 + 2.B(r)dr.dt - C(r)[r^2(d\theta^2 + \sin^2\theta d\varphi^2)] - D(r)dr^2 \tag{11}$$

Painlevé states that:

"'Whatever else the functions A, B, C, D of r that the experiment would lead us to adopt, it would still be possible to form invariant conditions, to be satisfied by the coefficients of the ds^2, when we replace r, θ, φ and t by functions of four totally arbitrary other variables."'

Then, he will constrain its general form by Einstein's equation to obtain a relativistic solution. Let us read how he explains that :

"'But Einstein wants, a priori, that these invariant conditions are partial derivatives of the second order of a special form, which is inspired by both the theories of Newtonian gravity in curvilinear coordinates, and the theory of the curvature of the ordinary surfaces. [3] *It is these fundamental restrictions and not the pure and simple truism of invariance, which among the ds^2 of form given by equation (11) leave only the following"':*

$$ds^2 = (1 - \frac{2\mu}{f(r)})(dt - \chi(r)dr)^2 - f^2(r)[d\theta^2 + \sin^2\theta d\varphi^2] - \frac{f'^2(r)}{1 - \frac{2\mu}{f(r)}}dr^2 \tag{12}$$

"'In this equation (12), μ is a constant and $f(r)$ and $\chi(r)$ are two arbitrary functions of r such that $\chi(r)$ tends to zero and $f'(r)$ (always positive) tends to 1 when r tends to infinity"'.

7.2 Painlevé proposes the most general form of solution to the problem in these coordinates allowing to cover the maximally extended spacetime

This equation (12) gives the most general relativistic form of the solution to the problem of the external field generated by a mass with spherical symmetry. It is straightforward to verify that whatever the functions $f(r)$

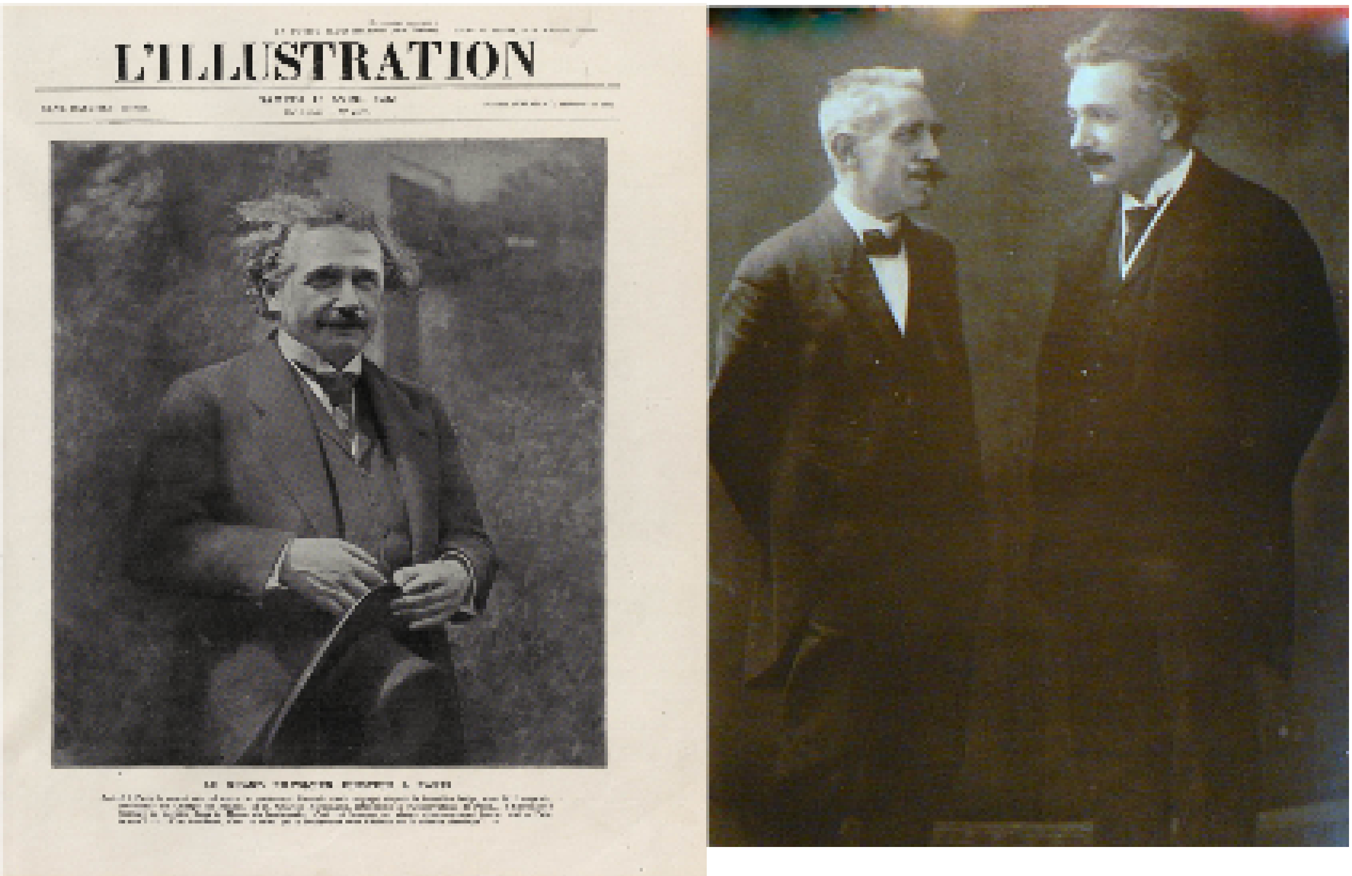
[3] Without saying it explicitly, Painlevé will constrain his form by the Einstein equation.

and $\chi(r)$ are, they satisfy the Einstein equation. For instance, if we set $\chi(r) = 0$ and $f(r) = r$, therefore $f'(r) = 1$, we get the Schwarzschild's metric and if we set $\chi(r) = \sqrt{(2M/r)}/(1 - 2M/r)$ and $f(r) = r$, this generic form gives the solution, (without the misprint), that Painlevé had proposed in his article of October 1921. It is straightforward to verify that this equation can generate all the forms, in these coordinates, including also some well-known forms, such that given by Eddington, Finkelstein and isotropics forms. Each form of the metric covers half of the maximally extension of spacetime, but they may be associated by pairs, like in the Painlevé form, differing only by the sign of the cross product $dr.dt$, for covering the full maximally extension of spacetime in this problem, i.e the four regions of spacetime, as it will described by the unified formalism given by Kruskal in 1960 [30]!

Painlevé therefore deduces directly from the symmetry of the problem, constrained by the Einstein's equations, a class of solutions where some of them are not singular on the horizon, and this more than 10 years before Lemaître. It is a totally unknown contribution that must be acknowledged.

8 DEBATES AT THE COLLÈGE DE FRANCE WITH EINSTEIN IN SPRING 1922

In November 1921, Painlevé wrote to Einstein for presenting his remarks on the Schwarzschild's solution [22] and announced his solution. He invited him to discuss it with him and his colleagues at the Academy of Sciences. In a letter of December 7, Einstein responds to Painlevé's critics and invitation, but having some commitments, he informed him that he will not be able to come in Paris very soon. [21] He will come in the spring of 1922 and will make a series of lecture-debates from March 31 to April 7 [25]. Charles Nordmann [14] will give an account of the discussions: ”Einstein exposes and discusses his theory” published in the ”‘Revue des Deux Mondes”’ [8]. Charles Nordmann begins by pointing out that Einstein's performance at the Collège de France, at the invitation of Paul Langevin, was an unprecedented event. In the United States, in London, Italy, where Einstein had been received in the preceding months, he just made ”‘ex cathedra ”‘lectures.



(a) Einstein in Paris-1922

(b) Einstein and Langevin

Figure 3: Einstein in Paris left, Einstein and Langevin right

Einstein's esteem for the French scientific school brilliantly represented by Langevin and his friends, including Painlevé, is reciprocal. Witnessed, a group of academicians, including Painlevé, which were willing to propose Einstein as a correspondent in a position that was going to be vacant soon, at the Academy of Sciences, for much to the chagrin of an influential member of this Academy who said that it was impossible to offer that to "the one who destroyed classical mechanics". In Paris, Einstein will make the effort to express himself in French and adopt a resolutely dialectical attitude, will intervene in a contradictory way with his interlocutors, let them debate between them, under his arbitration, in order to clarify the debate. The opening session on March 30th will be held in front of a large and enthusiastic public, the Collège de France having opened its doors to the men of sciences and also to students.

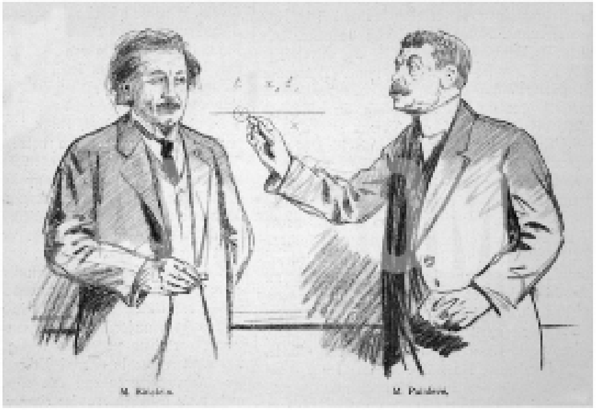
Figure 4: The crowd crowding the doors of the Collège de France for attending a conference of Einstein. Painlevé at the door is filtering the entrances(Gallica BnF image).

Other sessions, more technical, will follow with of a smaller, but more specialized in physics, audience such as the debate with Painlevé, in the presence of H. Becquerel, M. Brillouin [16], E. Cartan [17], T. De Donder, J. Hadamard, P. Langevin and C. Nordmann.

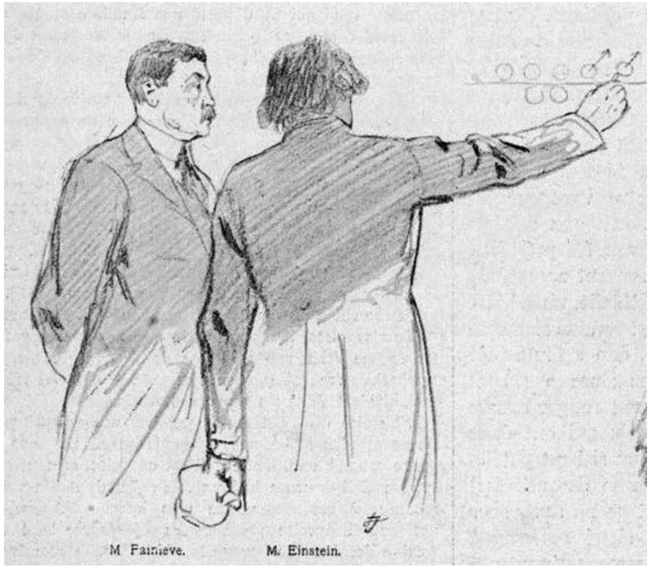
Apart from the Collège de France, Einstein, accompanied by Langevin will also hold a conference at the Société Astronomique de France [15]where he received a warm welcome.

9 THE PROBLEM OF THE HORIZON IN THE SOLUTION OF THE SINGLE BODY WITH SPHERICAL SYMMETRY

Here are excerpts from Charles Nordmann's narration of this small circle meeting: It was Mr. Hadamard, celestial mechanics professor at the Collège de France, who opens the debate with a question relating to the formula by which Einstein expresses the new law of universal gravitation. In this formula, under the simple form that Schwarzschild gave to it and that answers all the practical needs of astronomy, there is a certain term that Mr. Hadamard is very much concerned with. If the denominator of that term becomes null, meaning if this term becomes infinite, the formula no longer make sense or at least one could demand what is its physical



(a) Einstein and Painlevé, drawing in l'Illustration



(b) Painlevé and Einstein, drawing by Jonas

Figure 5: Drawings Einstein vs Painleve

meaning and how it could occur in nature. This is not the case of the Sun but it could be the case of a star that could be much more massive than him. Einstein does not hide the fact that this profound question is somewhat embarrassing to him."'If"'he says "'this term could effectively become null somewhere in the universe, then it would be an unimaginable disaster for the theory, and it is very difficult to say *a priori* what could occur physically, because the formula ceases to apply"'. Is this catastrophe, which Einstein, with humor, calls the "Hadamard catastrophe" possible and in this case what would be its physical effects.

Charles Nordmann intervenes then to give some precisions on very massive known stars, as Betelgeuse whose diameter is worth 300 Sun but which is far from satisfying the dreaded criterion. He reports that, according to the work of the English astronomer Eddington: when the mass of a star tends to increase more and more, because of the gravitational attraction, the inner temperature of this star increases significantly and the radiation produced tends to throw outward any new addition of matter. As a result, it seems to be in the nature of things, that an insurmountable limit exists for the mass of a star. This should protect us from the "Hadamard catastrophe" that should never occur to happen, because the conditions of existence of such stars could never be satisfied.

Einstein replied to me that he was not entirely satisfied by these calculations, that involve several hypothesis. He would much prefer an other argument which would avoid "'the misfortune that the Hadamard's catastrophe represents for the theory"'.

Effectively, in the following session, of April 7th, he brought up the result of a calculation he had made concerning this critical point which shows that if the volume increases indefinitely without increasing the density, which would be the case for a sphere of water, the pressure at the center of the mass would become infinite, and this would happen long before the conditions of Hadamard's catastrophe were met. Under these conditions in accordance with the theory of general relativity, the clocks are frozen and nothing can happen and therefore any change capable of bringing the "'Hadamard's catastrophe"' would become impossible. Mr. Hadamard, in these conditions declared himself satisfied and thought that it was impossible for this dreaded catastrophe to occur.

Painlevé takes this opportunity to ask Einstein some questions regarding his gravitational and similar formulas which now allow to express new phenomena (the advance of the perihelion of Mercury, the deviation of light by gravity) observed in the fields of celestial mechanics and optics. What followed was an extremely brilliant and sprightly debate on the physical meaning of the equations. Brillouin, unable to express himself in this tumult, leap to the blackboard and chalk in hand, inscribes his own contributions, this restoring the attention of a somewhat disturbed audience! Einstein, who listened silently, indifferent to the tempest, politely raised his hand, for asking the attention of the audience. This relaxed the atmosphere, and restored the silence. It only took a few minutes, for Einstein to convince the audience and reduce the main objections ..

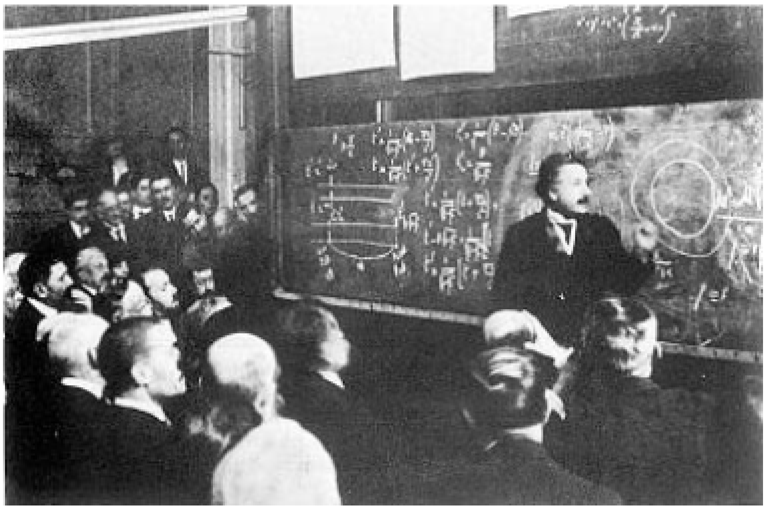


Figure 6: Einstein at the blackboard, in a session at the Collège de France. Painlevé is sitting at the left end of the blackboard

10 THE FORGOTTEN PAINLEVÉ SOLUTION

We see that the enthusiasm of the participants has generated some confusion in these debates, and that the problems on the horizon, that the form of Painlevé allowed to eliminate, have not been treated in depth, but evaded on the grounds that the formation of a horizon was not physically possible. This form, whose non-quadratic term implies an orientation of the spacetime, necessary in this type of coordinates, for being non-singular on the horizon since this one behaves like a unidirectional membrane (can be only crossed through inwards), was not understood at the time. It does, however, offer a natural description of the spherical symmetrical solution of the field of a single body, and reveals the deep symmetries and certain attributes that it shares with Newtonian mechanics without being confused with it. This opportunity has been missed and the major contribution of Painlevé will be forgotten in history for about 80 years.

THE FORM OF PAINLEVÉ TODAY

This form of metrics for this space-time, the first non-singular on "the horizon", presents some Newtonian characters, which probably inspired the choice of Painlevé. The free fall radial equation (zero velocity to infinity) obeys to the same equations as those of Newtonian mechanics. A metric is generally associated to a "fiducial" observer (in free fall from zero velocity at infinity) who, under these conditions, is in a quasi-Newtonian situation. Nevertheless the solution remains relativistic and should not be confused with the Newtonian solution. Forgotten for 80 years, this metric has recently aroused interest, as evidenced by the articles by Martel and Poisson ([16], 2000), Taylor and Wheeler (2000) [28] which call it the "'rain frame"', Doran (2000) [29] and Hamilton and Lisle ([17], 2006), all emphasizing and exploiting this pseudo-Newtonian character to propose simple and original descriptions of this spacetime especially in Cartesian coordinates where it can be written [24]:

$$ds^2 = \eta_{\mu\nu}[(dx^\mu - \beta^\mu dx^0)(dx^\nu - \beta^\nu dx^0)] \tag{13}$$

Where β, with $r = \sqrt{x^2 + y^2 + z^2}$, is called the shift 4-vector which is defined by:

$$\beta^\mu = \sqrt{\frac{2GM}{r}}[0, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}] \tag{14}$$

Moreover, under this form the pseudo-tensor describing the gravitation, in the Landau-Lifchitz theory of gravitation (a field in the Minkowski spacetime) [32], is null which is a non trivial property[31].

11 DEBATES AT THE ACADEMY OF SCIENCES

As we have said, the Academy of Sciences had ignored the general relativity until 1921, when Einstein won the Nobel Prize in physics which will be awarded in 1922. If this prize is not awarded for general relativity, which

Figure 7: Reception of Einstein at the École Polytechnique. We recognize Langevin, on the right of Einstein, and Marie Curie on the left, who turns his head to the lens. photo ESPCI all righs reserved

shows that this theory was far from being accepted at the time, it gives to him a notoriety that the Academy of Sciences can not ignore. By its constitution, the Academy of Sciences is a rather conservative institution. Some members of the Academy, among the most active, are hostile to the theory of general relativity which they consider as destroying the classical mechanics. This current, led by Jean Le Roux, author of a multitude of notes aimed to disqualify the theory of Einstein, is dominant at the beginning. But soon, it will emerge a current, led by M. Brillouin, of scientists seduced by this new and constructive approach to gravitation for exploring the universe, differently. The contribution of Painlevé, beginning in October 1921, after a virulent critic of general relativity by M. Le Roux, was intended to "moderate" the debate since it proposed to make a comparative critical study of the two theories for his colleagues quite baffled by this new theory. If, per his scientific background, it was difficult for him to be fully objective, his attempt has been highly constructive and has contributed to initiate the enlightening debate that followed at the Académie of Sciences.

11.1 The controversy initiated and led by Le Roux

The quite fierce controversy can be illustrated by some examples of debates between the leaders of the two sides. The beginning of the first article of Jean Le Roux (1863-1929, professor at the Faculty of Sciences of Rennes) in May 1921 [8] will give a flavor on Le Roux's specious arguments for disqualifying the general relativity.

Le Roux 1921: The discovery of a law of gravitation explaining the movement of Mercury's perihelion has been regarded as a striking confirmation of the theory of relativity. A judicious criticism states that this result has been obtained, while using the theory of relativity, but that it is not a consequence of it and does not even constitute an argument in its favor.

Le Roux will continue his destructive criticism in many other articles, for example in November 1922 [9]:

Le Roux 1922: The results provided by Einstein's theory of gravitation seemed, at first, to agree remarkably with observation especially in the case of the secular motion of Mercury's perihelion. However, to arrive at this conclusion, we are obliged to admit that the disturbances due to the mutual actions of the planets preserve in Einstein's theory the same values as in classical mechanics. If the disturbances are removed, the concordance disappears. But it happens that Einstein's fundamental hypothesis is incompatible with the existence of mutual actions and perturbations, such as we consider them in classical mechanics.

Therefore he will conclude his article as follows:

Le Roux 1922, conclusion: The confrontation with experience in the particular case of the motion of Mercury gives rise to the following observations. The secular advance is 574". Newton's theory of disturbance provides a satisfactory explanation up to a maximum limit of 536" with an unexplained value of 38". In Einstein's theory, the movement deduced from ds^2 calculated by Schwarzschild would give Mercury a secular advance of 42"9. But since this theory excludes disturbances due to mutual actions, an unexplained value of 531" remains. This is the brutal result! Therefore, we must note that Einstein's theory, in its current state, does not allow us to explain or predict, even with the most crude approximation, the secular movement of Mercury.

Marcel Brillouin, on November 1922 ([10]) will reply to the arguments of J. Le Roux as follows:

Brillouin reply Everyone knows that Einstein's theory of gravitation includes Newton's theory as first approximation. The note of M. Le Roux shows that it is nevertheless useful to recall it. The ten potentials of Einstein $g_{\mu\nu}$ contain not only the coordinates x^μ of the point of the space around which they define the ds^2, but also, contrary to the singular affirmation of M. Le Roux (P. 810, lines 21-27), the coordinates χ of all the singular points of space, i.e all the points where there is an attracting mass, fixed or mobile, as well as the value of each one of these masses. When these masses are mobile, per only their mutual influences, we are obliged to study altogether the motions of all these masses, that is to say, to treat the problem of n bodies either in its rigorous form, when we will know exactly how to form the $g_{\mu\nu}$ of a space containing n moving bodies, as a function of the four coordinates χ of each of these bodies, or approximately in the manner of the disturbance problem. At first order, because of the high value of the velocity of light, the only potential that differs from that of a Euclidean space is the coefficient dt^2, g_{44}, which at this order of approximation, obeys the equation of Poisson and therefore is formed as in Newtonian theory and provides all the classical disturbances. The criticisms of M. Le Roux are therefore completely unfounded.

This fierce debate will last until the beginning of 1924, when the balance between pros and cons will change. The position of Mr. Le Roux begins to look more and more like a sterile rearguard battle. Painlevé, recalled by his political career left the debate after his article of May 1, 1922, following the visit of Einstein to the Collège of France.

11.2 The remarkable contributions that emerged from this debate

Regardless of the innovative contributions of Painlevé that have been described in this document, other contributions at the Académie of Sciences are worth to be cited.

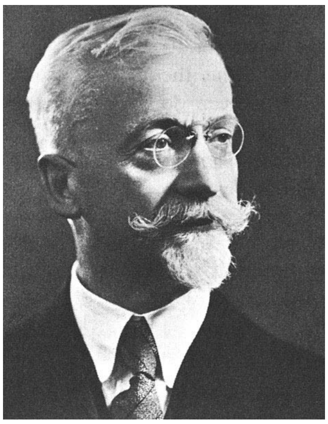


Figure 8: The Mathematician Elie Cartan (1869- 1951), who brings major contributions to mathematics and general relativity . Photo WikiCommons

Cartan 1922: Elie Cartan, in a note of March 27, 1922 "On generalized conformal spaces and the Optical Universe" [17], defines the principal null directions of a space-time. After recalling the properties of a conformal space, Cartan writes:

The case $n = 4$ is particularly important. Let us call "'Einstein Optical Universe"' the space(time) defined by setting $ds^2 = 0$ in the Einstein Universe. The geometrical properties of this optical Universe describes the null geodesics (propagation of light). The rotation curvature of this Universe is defined at each point by ten scalar quantities, or by a ternary quadratic form with complex coefficients, that a change of the reference system transforms by an orthogonal substitution. From a geometrical point of view the following property are worth to be pointed out. There exists in each point A four optical directions, (such that $ds^2 = 0$), privileged. They are characterized by the property that if AA' is one of them, it is preserved by the displacement associated with an elementary parallelogram admitting AA' as side and any other optical direction coming from A. In the case of a only attractive mass (ds^2 of Schwarzschild), these four privileged optical directions are reduced to two (double): The two light rays which go to the center of attraction or would come from it.

Sauger1922: The article by Maurice Sauger [18] (April 1922, [12]) "On a remarkable coincidence in the theory of relativity" is interesting for several reasons. On the one hand it can help to understand what inspired Painlevé in the development of its original form, even if here Sauger refers to the Schwarzschild form. On the other hand, it clarifies the links between this solution and that of the Minkowski's space-time and gives a clear interpretation of the phenomenology of radial spectral shifts, as formally calculated, whose results may look paradoxical. The author very simply constructs the Schwarzschild form from the special relativity and Newtonian mechanics simply by noting that the velocity v of a body in free fall from infinity is $v = -\sqrt{(2GM/r)}$, as compared to a (static) observer at infinity.

Chazy 1922: The article by Jean Chazy [19] (May 1922, [13]) "On the astronomical verifications of the theory of relativity" is the first to rigorously establish the form of the Schwarzschild metric with a cosmological constant by deriving directly the metric from the equation of Einstein with a cosmological constant. It is a very original contribution, totally forgotten, exposing a solution that will be also demonstrated by Lemaitre ten years later.

11.3 What happened to these remarkable contributions?

All these innovative and fundamental contributions resulting from the dialectical debate that took place during this period have fallen into oblivion. Their importance was not obvious to the scientific community, likely because they resulted more from a mathematical formal work about the equations of the theory than from a deep understanding of the underlying physics. This shows that at that time, even among the brightest minds like Einstein, the perception of the physical implications of the theory of general relativity was far from being clear, as evidenced by a certain lack of consideration for the Painlevé's proposal, which allowed to formally solve the problem of the singularity on the horizon which was baffling so much the Einsteinians. It is the way the science works, but it is a pity that this opportunity, for the french scientific community to participate to the development of the general relativity, was spoiled. This would undoubtedly have changed the course of their participation in the development of this theory and would have given it a significant place in the construction of this pillar of science of the twentieth century.

12 Conclusion

Painlevé acknowledges that if Einstein's geometrical formalism for the gravitation was a daring approach and seduced him, he was not ready to abandon the whole edifice of classical mechanics in favor of general relativity. The geometrical method that he developed for the Newtonian mechanics for comparing the two theories was very effective, as we have pointed out, since it induces attributes, well beyond its objectives, such as the concept of a proper time in Newtonian mechanics! Reluctant to adopt the general relativity, he was nevertheless a good mathematician and had assimilated the mathematical background of relativity, as his contribution shows. This detachment from the physical phenomenology of the theory allowed him to approach the problem without any physical constraints which, because they were poorly understood at that time, were more inconvenient than helpful. The brilliant contributions of his colleagues from the Academy of Sciences probably fit in the same context. The problem of the horizon is edifying, since the form established by Painlevé shows that it is a false problem, whereas Einstein, in his lectures at the Collège de France, invokes peripheral arguments to show that this case can not physically occur. This attests of the complementarity between the "discoverers of theories" and mathematicians clarifying these discovered theories, the two fields demanding different qualities. The sucessful development of science calls for a harmonious cooperation of both.

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