Introduction: The Tsiolkovski equation

The classical Tsiolkovsky equation gives the increase in speed ΔV of a rocket during the phase of propulsion. Let us stress that has been established in the co-moving frame of the rocket by integrating a local equation of momentum conservation. In Newtonian mechanics, time and space being universal, this result was universal, unlike in relativity where time and space depend on observers.

Relativistic Tsiolkovski equation
The Rocket frame

Established in the co-moving frame of the rocket, its dynamic parameter is the astronaut's proper time, measured by his watch.
The increase of the "proper" velocity of the rocket ΔV is measured by integrating the data of inertial devices such as gyroscopes which will define this parameter among all the trajectory's parameters.

The ground frame

For the observer at rest, on the ground, it is with the time t, given by his watch (its proper time is different of that in the rocket), and the space x, given by a previous measurement of space of his frame of reference denoted R, that the observer will evaluate the speed v = dx/dt, of the rocket, which is tangent to the curve x(t) describing its trajectory from the ground.
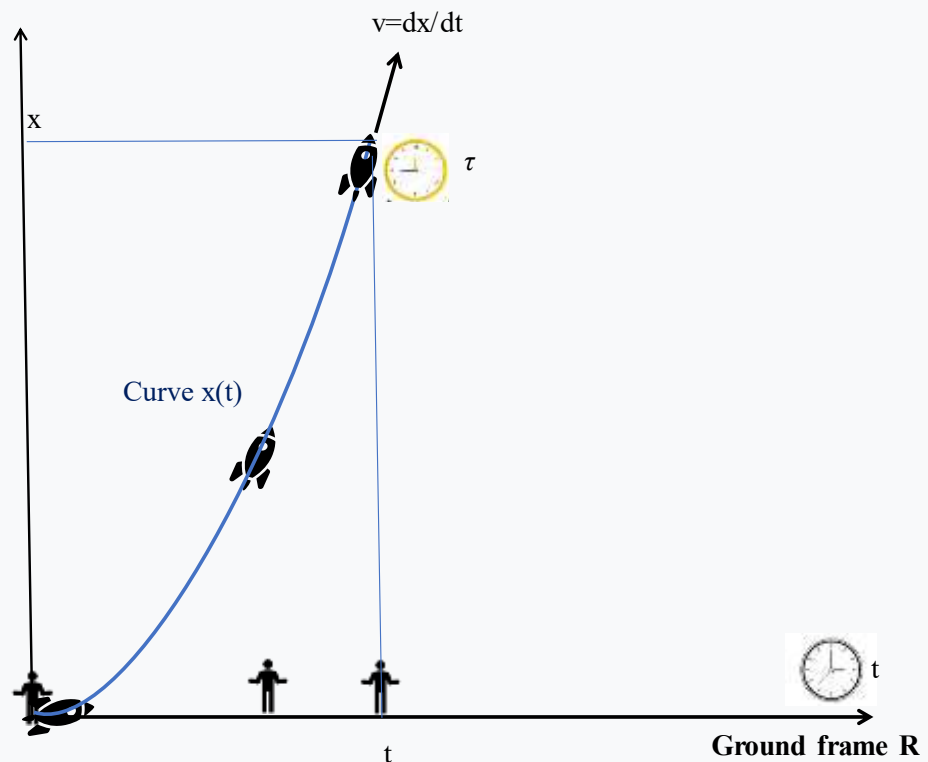
Formalism for accelerate motion

For an accelerated motion, the general relationship between these parameters is given by a relationship like:

$$t = a^{-1} sinh(a.\tau) , x = = a^{-1} cosh (a.\tau)   (1)$$

where t, x are the coordinates of the ground frame denoted R, a is the proper acceleration and τ is the proper time of the observer in the rocket.

Equation (1) is used in Rindler coordinates for a constant acceleration. The rocket has an acceleration which is not constant during propulsion step, so we have to replace a by $a(\tau)$. As this equation is valid locally, for the increase of velocity we will then get:

$$a.\tau = \int_{\tau_0}^{\tau_f} a(\tau)d\tau$$

Physically, the value of this integral will be given by integrating the data provided by the inertial devices, exactly as in the classic Tsiolkovski equation, this providing obviously, the same result. We do not have to solve this integral as we know its result.

$$a.\tau = \int_{\tau_0}^{\tau_f} a(\tau)d\tau = \Delta V = v_e \ln(m_i / m_f)$$

The rocket in the ground frame

Consider this equation (1), at the end of the propulsion.
The speed in the ground frame R is:

$$v = dx/dt:$$

let us differentiate (1)

$$dx = \sinh(a\tau)d\tau, \quad dt = \cosh(a\tau)d\tau$$

this gives

$$v = dx/dt = \tanh(a\tau) \quad (2.1)$$

where

$$a\tau = V = v_e \ln(m_i/m_f) \quad (2.2)$$

V is the speed acquired in the frame of reference of the rocket, obtained by integrating the proper acceleration during the proper time τ of propulsion (principle of inertial navigation).
By plugging (2.2) in (2.1), the velocity in R is:

$$v = \tanh(v_e \ln(m_i/m_f)) \quad (3)$$
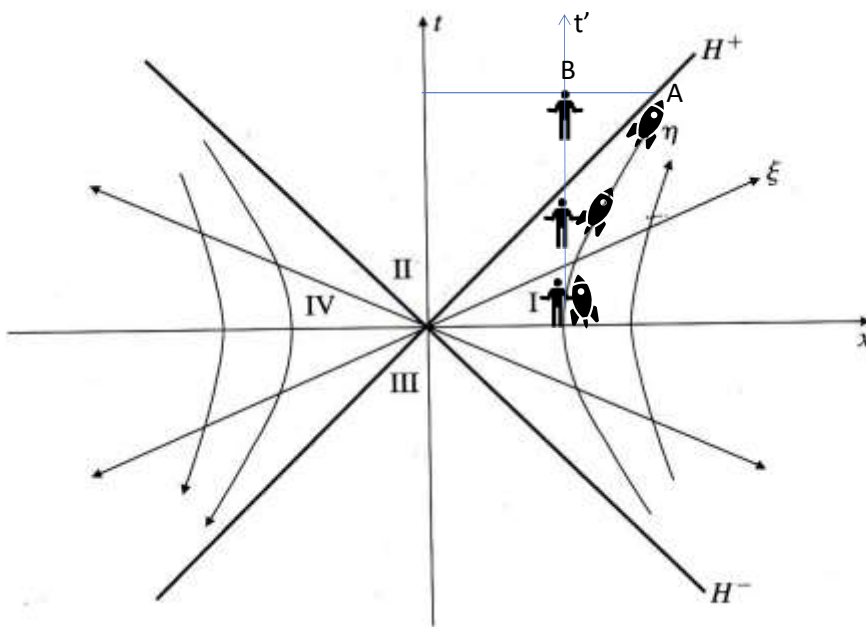
This is the relativistic Tsiolkovski equation

Convergence with classic Tsiolkovski equation when v/c << 1

By developing tanh(x) in Taylor's series, one shows that if v << c, we get:

$$v/c \approx (v_e/c) \ln(m_i/m_f)$$

which is the classical Tsiolkovsky equation.

Rindler formalism and horizon

Let us use Rindler coordinates, which are well suited to this type of phenomenology.

When the rocket is at A, it is beyond the horizon for observer B who has remained stationary in the initial frame of reference.

Another demonstration, chapter "Special relativity" on the link:
https://en.wikipedia.org/wiki/Tsiolkovsky_rocket_equation