Precession of geodesics in space section of Schwarzschild's spacetime

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Summary The precession, resulting from the 4D spacetime curvature, of spacetime geodesics in Schwarzschild's spacetime, is well-known topic.[1]. In this article we will derive the precession of a geodesic in the space section of the Schwarzschild's spacetime. This precession which, unlike the precession in spacetime, only depends on the ratio [2] M/l , is defined by an original infinite polynomial of even powers of M/l, this providing a real precession, even for imaginary values of l. In the conclusion, we will discuss how this solution may enlighten the understanding of the precession phenomenology.

1 Space geodesics in Schwarzschild's metric

In the three dimensional space section of Schwarzschild's spacetime where $d\sigma^2$ is the metric line element, we can also to define the precession of a geodesic in this space section by a $r(\varphi)$ function, with an angular momentum defined by $l = r^2 d\varphi/d\sigma$. [3]

Original Schwarzschild's metric is recalled in equation (1) below.

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) = -(1 - \frac{2GM}{r})dt^2 + d\sigma^2 \tag{1}$$

Where,

$$d\sigma^2 = +\frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 \ + \sin^2\theta d\varphi^2) = \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\varphi^2, \text{ for } \theta = \frac{\pi}{2} \tag{2}$$

In the last part of the equation , we set $\theta = \pi/2$, which is allowed, per the spherical symmetry, without loss of generality.

2 Analytic method for the precession of planets in Schwarzschild's spacetime

2.1 Interest of geodesics in a space section of spacetime

General relativity is a geometrical theory of the gravitation in spacetime. Physical geodesics are timelike or null geodesics. [4] Unlike geodesics in spacetime which do not depend on the coordinates, geodesics in a space section of spacetime depend on the coordinates. So, what would be the interest of such geodesics? This interest is motivated by an original proposal of Painlevé [3] describing the Schwarzschild's spacetime geodesic as the geodesic of the Schwarzchild space section multiplied by a conformal factor. This shows that the conformal structure of the space section and that of the spacetime are the same. This is a quite surprising property.

Therefore, as one defines often the spacetime geodesic by using the Schwarzschild coordinates, geodesics in its space section would provide a complementary information for describing the geometrical space structure of a space section which is more complex than it appears at a first look, see [5], figure 1.

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[1] For analytic solution solving the spacetime equation, see for instance [4], chapter 19.

[2] M is the mass of the central body, l is the angular momentum of the space geodesic

[3] In 4D spacetime, the affine (dynamic) parameter on a timelike geodesic is the proper time, a timelike parameter. In space the affine parameter on the spacelike geodesic, is a spacelike parameter therefore the dimension of $L = r^2 d\varphi/d\tau$ will be a square length divided by a time while that of $l = r^2 d\varphi/d\sigma$ will be a length.

[4] One can also define spacelike geodesics in spacetime but there are not considered to be physical.

2.2 The geodesic equation in space

We will use the well-known method for getting a general solution for the geodesic equation in space and then, we will use it for solving the problem of the precession of geodesics in space. Dividing second part of equation (2) by $d\sigma^2$ yields:

$$1 = \frac{dr^2}{d\sigma^2(1-\frac{2GM}{r})} + r^2\frac{d\varphi^2}{d\sigma^2} \Rightarrow \frac{dr^2}{d\sigma^2} = (1-\frac{2GM}{r})(1-r^2\frac{d\varphi^2}{d\sigma^2}) \tag{3}$$

$$\frac{dr^2}{d\sigma^2} = (1-\frac{2GM}{r})(1-\frac{l^2}{r^2}) \tag{4}$$

Equation (4), valid only on a geodesic, is equation (3) with $l = r^2 d\varphi/d\sigma$ which is the conserved angular momentum, on the spatial geodesic.[5]

By multiplying equation (4) by $(d\sigma/d\varphi)^2 = r^4/l^2$, we get:

$$\frac{dr^2}{d\varphi^2} = (1-\frac{2GM}{r})(\frac{r^4}{l^2}-r^2) \Rightarrow d\varphi = \frac{\pm dr}{\sqrt{-r^2(1-\frac{2GM}{r})(1-\frac{r^2}{l^2})}} \tag{5}$$

Let us set:

$$u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \Rightarrow dr = -\frac{du}{u^2} \tag{6}$$

By inserting it, in equation (5), we get:

$$\frac{du^2}{u^4 d\varphi^2} = (1-2GMu)(\frac{1}{u^4})(\frac{1}{l^2}-u^2) \Rightarrow d\varphi = \frac{\pm du}{\sqrt{(1-2GMu)(\frac{1}{l^2}-u^2)}} \tag{7}$$

By defining an angle θ, a parameter A^2 and a constant K, such as:

$$\theta = \arcsin\sqrt{\frac{1+lu}{2}} \Rightarrow \sin^2\theta = \frac{1+lu}{2}, A^2 = \frac{4GM}{2GM+l}, K = \sqrt{\frac{l}{l+2GM}} \tag{8}$$

Equation (7) can be written:[6]

$$\frac{d\varphi}{2} = K\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} \Rightarrow \frac{\varphi(\psi,A^2)}{2} = K\int_{\theta=0}^{\theta=\psi}\frac{d\theta}{\sqrt{1-A^2\sin^2\theta}} = K Elliptic_F(\psi,A^2) \tag{9}$$

Inserting the values of θ, K and A^2 defined in equation (8) yields:

$$\frac{\varphi}{2} = \sqrt{\frac{l}{l+2GM}} Elliptic_F[\arcsin(\sqrt{\frac{1}{2}(1+lu)}), \frac{4GM}{l+2GM}] \tag{10}$$

$Elliptic_F(\psi,A^2)$ or $F(\psi,A^2)$ in a short notation is the integral described in equation (9). This integral, called elliptic integral of the first kind, includes an argument A^2 called the parameter, A is called the modulus.[7] The parameter $\psi = \theta(u)$, called the amplitude, is, as shown in equation (9), the upper limit of integration of the angle θ defined in equation (8).

Returning to $r = 1/u$, we get:

$$\frac{\varphi}{2} = \sqrt{\frac{l}{l+2GM}} Elliptic_F[\arcsin(\sqrt{\frac{1}{2}(1+\frac{l}{r})}), \frac{4GM}{l+2GM}] \tag{11}$$

[5] This "'constant of motion"' l exists as the metric $d\sigma^2$ does not depend on φ.

[6] By using the definition of θ, A^2 and K in equation (8), it is straightforward to verify that $du^2/[(1-2GMu)(-u^2+1/l^2)] = (2Kd\theta)^2/(1-A^2\sin^2\theta)$ whose integral is an elliptic integral of first kind.The binary operator $\pm$, in the last term of equation (7), is related to the orientation of the geodesic as there are two possible orientations. For the integration, we can select one direction, without loss of generality, that we will associate to the sign $+$.

[7] There are several formal notations, this being quite confusing. For instance, it is denoted $Elliptic_F(\psi, A)$ in WolframMathWorld but, in both notations, it is A^2 which is used in the computation of the integral. It is just two notations for the same object. This remark will also apply to the $Elliptic_K$ integral and Jacobi-Amp function, that we will use further.

2.3 Elliptic-K

$$K(A^2) = Elliptic_K(A^2) = Elliptic_F(\frac{\pi}{2}, A^2) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - A^2 \sin^2\theta}} \tag{12}$$

$Elliptic_K(A^2)$, also called $K(A^2)$ in a short notation, is a special case of $Elliptic_F$, where the upper limit ψ is equal to $\pi/2$. It is the complete elliptic integral of the first kind of Legendre and therefore has only one parameter (A^2). There exists an analytic definition of the integral $Elliptic_K(A^2)$ by an infinite polynomial of powers of A^2. This is this polynomial definition, given by equation (19), that we will use further, for calculating the precession.

With the values of K and A^2, defined in equation (8), by using the definition of $Elliptic_K(A^2)$, the equation (10) for $\psi = \pi/2$, can be written:

$$\frac{\varphi(\psi = \pi/2, A^2)}{2} = \sqrt{\frac{l}{l + 2GM}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - A^2 \sin^2\theta}} = \sqrt{\frac{l}{l + 2GM}} Elliptic_K(\frac{4GM}{l + 2GM}) \tag{13}$$

Equation(13) that we will use for computing the precession, defines $\varphi/2$ and not φ therefore, we will have to multiply it by two for getting the result. Equation (13) shows that the solution in space only depends on the parameters M and l.

2.4 Jacobi-Amp-function

The $Elliptic_F$ integral defined in equation (10) $\varphi/2K = F(\psi, A^2)$ has an inverse function called Jacobi-Amplitude function noted $am(\varphi/2K, A^2)$, such that $\psi = am(\varphi/2K, A^2)$. So per the definition of $\sin(\psi)$, where ψ is the upper limit of θ, and K in equation (8) we get [8]:

$$\sin^2\psi = \frac{1 + lu}{2} = sn^2(\frac{\varphi}{2K}, A^2) \Rightarrow u = \frac{1}{l}(-1 + 2sn^2(\frac{\varphi}{2K}, A^2)) \tag{14}$$

This gives the function $u(\varphi)$. The function $r(\varphi)$ can be deduced by using the relation $r = 1/u$.

2.5 These elliptic integrals define the precession

$Ellipic_F, Elliptic_K$ and their inverse integrals exhibit two angles ψ and φ. Equation (13) shows that when θ varies from *0* to $\psi = \pi/2$, $\varphi/2K$ varies from *0* to $Elliptic_K(A^2)$. So for a half-pseudo-orbit where θ varies from *0* to π [9]:

$$\Delta(\frac{\varphi}{2}) = 2K Elliptic_K(A^2) - \pi \tag{15}$$

And equation (16) below will be the equation to be used for solving the problem for n full orbits.

$$\Delta\varphi = 4n(2K Elliptic_K(A^2) - \pi) \tag{16}$$

2.6 Class of equivalence

The equation (16) only depends on parameters K and A^2. This equation will provide an exact solution to the problem, if we know these parameters on the spacelike geodesic. By posing $2GM/c^2l = k^2$, parameters K and A^2, defined in equation (8), can be written:

$$K = (1 + \frac{2GM}{c^2 l})^{-1/2} = (1 + k^2)^{-1/2}, A^2 = (\frac{2GM}{c^2 l})(\frac{2}{1 + \frac{2GM}{c^2 l}}) = (2k^2/(1 + k^2)) \tag{17}$$

Therefore, equation (16) will only depend on the dimensionless parameter k^2. We will expect a solution as a function of k^2.

This parameter k^2 defines a class of spatial solutions.

[8] In Jacobi elliptic functions, $sn(\varphi/2K, A^2) = sin(am(\varphi/2K, A^2))$, see WolframMathWorld, Jacobi elliptic functions.

[9] A half-orbit defines the dynamic as we assume the symmetry of the orbit for the precession.

3 Solution for the precession in space

3.1 General solution

By inserting the definition of A^2 and K, given in equation (17), in equation (13), giving the formal general solution, for $\psi = \pi/2$, we get:

$$\frac{\varphi}{2} = \sqrt{\frac{1}{1+k^2}} Elliptic_K(\frac{2k^2}{k^2+1}) \tag{18}$$

The integral $Elliptick_K(k)$ can be represented by an infinite polynomial: [10]

$$Elliptic_K(\frac{2k^2}{k^2+1}) = \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n!)}{2^{2n}(n!)^2}]^2(\frac{2k^2}{k^2+1})^n \equiv \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(\frac{2k^2}{k^2+1})^n \tag{19}$$

where n!! denotes the semi-factorial. By using equation(19), equation (18) becomes:

$$\frac{\varphi}{2} = \sqrt{\frac{1}{1+k^2}}(\frac{\pi}{2})\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(\frac{2k^2}{1+k^2})^n = \frac{\pi}{2}\sum_{n=0}^{n=\infty}[\frac{(2n-1)!!}{(2n)!!}]^2(2k^2)^n(1+k^2)^{-\frac{1}{2}(2n+1)} \tag{20}$$

We replaced $-(n+1/2)$ by $-(1/2)(2n+1)$ which will be more convenient. Let us recall that $(1+k^2)^{-\frac{1}{2}(2n+1)}$ can also be developed in an infinite polynomial, as defined below:

$$(1+k^2)^\alpha = 1 + \sum_{j=1}^{j=\infty}\frac{(\alpha)(\alpha-1)..(\alpha-j+1)}{j!}(k^2)^j \tag{21}$$

For $-1 < k^2 < 1$ and where α is a real number, with $\alpha = -(1/2)(2n+1)$ in our problem.
By using these formulas, equation (20) becomes: [11]

$$\frac{\varphi}{2} = \frac{\pi}{2}\sum_{n=0}^{n=\infty}(\frac{(2n-1)!!}{(2n)!!})^2(2k^2)^n(1+\sum_{j=1}^{j=\infty}\frac{(2n+1)(2n+3)..(2n+1+2(j-1))}{(-2^j)j!}(k^2)^j) \tag{22}$$

The product of these two infinite polynomials is an infinite polynomial. For defining this polynomial we have to calculate each coefficient B_n of $(k^2)^n$ which, in the infinite polynomial defined in equation (22), will be the sum of the product of coefficients of the terms$(k^2)^i$ of the first polynomial with the coefficients of the terms $(k^2)^{n-i}$ of the second polynomial.
The result of this operation will define an infinite polynomial $P(k^2)$

$$P(k^2) = \frac{\pi}{2}\sum_{n=0}^{n=\infty}B_n(k^2)^n \tag{23}$$

Where in B_n, as the first factor of the sum is related to $(k^2)^i$, the second factor should be related to $(k^2)^{n-i}$, therefore in equation (22), we must set $n = i$, $j = (n-i)$. This yields: $(2i+1)(2i+3)..(2i+1+(2(n-i-1))) = (2i+1)(2i+3)..(2n-1)) = (2n-1)!!/(2i-1)!!$. Therefore B_n can be written:

$$B_n(k^2)^n = \sum_{i=0}^{i=n}(\frac{(2i-1)!!}{2i!!})^2(2^i(k^2)^i)(\frac{(2n-1)!!}{(2i-1)!!(n-i)!(-2^{n-i})})(k^2)^{n-i} \tag{24}$$

By simplifying by $(2i-1)!!$ and using $(2i-1)!! = 2i!/(i!2^i)$, $(2i)!! = i!2^i$, $2i!/i!^2 = 2i!/(i!(2i-i)!) = \binom{2i}{i}$, $n!/i!(n-i)! = \binom{n}{i}$, this equation yields:

$$B_n = \frac{2n!}{n!^2}(2^{-2n})\sum_{i=0}^{i=n}\frac{(-1)^{n-i}}{2^i}\binom{n}{i}\binom{2i}{i} = \frac{2n!}{n!^2}(-1^n)(2^{-2n})\sum_{i=0}^{i=n}(-\frac{1}{2})^i\binom{n}{i}\binom{2i}{i} \tag{25}$$

[10] $http://mathworld.wolfram.com/CompleteEllipticIntegraloftheFirstKind.html$, equation (2). In terms of the Gauss hypergeometric function, $Elliptic_K = (\pi/2)_2F_1(1/2, 1/2, 1, 2k^2/(1+k^2))$. Let us recall that the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ is a solution of the second order homogeneous differential equation $z(1-z)d^2y/dz^2 + [c-(a+b+1)z]dy/dz - aby = 0$.

[11] In the second sum, we will separate the factor 1/2 and the sign $-$ from the formula and will gather them in the factor $1/(-2^j)$ for simplifying the calculation.

3.2 The polynomial includes only even powers of (k^2)

We will demonstrate that equation (25) giving B_n is a hypergeometric series. Such series is defined by using the Gauss hypergeometric function ${}_2F_1(a;b;c,d)$. [12]

Let us set:

$$A(n) = \frac{2n!}{n!^2}(-1^n)(2^{-2n}) \tag{26}$$

Per the formal definition of the Gauss hypergeometric function:

$${}_2F_1(a;b;c,d) = \sum_{i=0}^{i=n} \frac{(a)_i(b)_i}{(c)_i}\frac{(d)^i}{i!} \tag{27}$$

Where the notation $(a)_i = a(a+1)..(a+i-1)$ is the Pochhammer symbol. If we set $a = -n$, $b = 1/2$, $c = 1$, $d = 2$, we get:

$(a)_i = (-n)(-n+1)..(-n+i-1) = (-1)^i(n)(n-1)..(n-i+1) = (-1)^i n!/(n-i)!$, $(b)_i = (1/2)(3/2)...(2i-1)/2 = (1/2)^i(2i-1)!! = (1/2)^i 2i!/(2^i i!)$, $(c)_i = (1)(2)...(i) = i!$, $d^i = 2^i$

Inserting these values in equation (27) yields:

$${}_2F_1(-n;1/2;1,2) = \sum_{i=0}^{i=n}(\frac{(-1)^i(n!)}{(n-i)!})((\frac{1}{2})^i\frac{2i!}{2^i i!})(\frac{1}{i!})(\frac{2^i}{i!}) = \sum_{i=0}^{i=n}(-\frac{1}{2})^i\binom{n}{i}\binom{2i}{i}) = \frac{B_n}{A(n)} \tag{28}$$

In the second sum of the equation above we used the relations: $n!/((n-i)!i!) = \binom{n}{i}$ and $2i!/(i!i!) = \binom{2i}{i}$.

This is the result that we expected!

For demonstrating that all terms $B_{2m+1}(k^2)^{2m+1}$ vanish, we need to use equation (16) of mathworld.wolfram-HypergeometricFunction, which gives an integral defining the hypergeometric function.

$${}_2F_1(a;b;c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}du. \tag{29}$$

By calculating this integral with our parameters $(a = -n, b = 1/2, c = 1, z = 2)$, we get:

$$\int_0^1 u^{-1/2}(1-u)^{-1/2}(1-2u)^n du = \int_0^{0.5} u^{-1/2}(1-u)^{-1/2}(1-2u)^n du + \int_{0.5}^1 u^{-1/2}(1-u)^{-1/2}(1-2u)^n du \tag{30}$$

Let us define two values of u, $(0 \leq u \leq 1)$, u_1 and u_2 such that $u_1 = 1/2 + a$ and $u_2 = 1/2 - a$, where $a \leq (1/2)$. We get $u_1^{-1/2}(1-u_1)^{-1/2} = [(1/2+a)(1/2-a)]^{-1/2} = u_2^{-1/2}(1-u_2)^{-1/2} = [(1/2-a)(1/2+a)]^{-1/2}$ (symmetry around 1/2).

In $(1-2u)$, for $u = u_1$ we get $-2a$ and for $u = u_2$, we get $2a$. This, raised to power n, will give $(u_1)^n = (-2a)^n$ and $(u_2)^n = (2a)^n$ which are equal when n is even and are opposite when n is odd.

Therefore, as exhibited by equation(30), where the integral is split in two parts (from 0 to 1/2 and from 1/2 to 1), when n is odd $(n = 2m+1)$, the two parts are opposite, the integral vanishes and when n is even $(n = 2m)$ the two parts are equal, this integral does not vanish.

Therefore:

$$P(k^2) = \sum_{n=0}^{n=\infty} B_{2n}(k^2)^{2n} \tag{31}$$

includes only even powers of k^2.

$$\frac{\varphi}{2} = K.Elliptic_K(A^2) = P(k^2) = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(k^4)^n = -\frac{\pi}{2}\sum_{n=0}^{n=\infty} B_{2n}(\frac{(2GM)^2}{(cL)^2})^n \tag{32}$$

The major interest of this polynomial form, describing the precession, is to show that, even for an imaginary value of the angular momentum, a real number for φ and therefore for the precession is expected! This property was not revealed by equation (13), as in case of imaginary value of l, we have the product of a square root of a complex number by an elliptic integral with complex arguments, whose form is not analytic, and whose tabulated numerical complex value is not exact, this spoiling the result.

[12] See mathworld.wolfram-hypergeometricFunction, equation (8) for the form of the series generated by such function, and the Pochhammer notation symbol $(a)_n$.

This is important because, per the form of the metric described in equations (1) and (2), if we assume that the angular momentum, on the timelike geodesic in spacetime, is real, that on the spacelike geodesic in space will be imaginary. Per this property, as both precession in spacetime and space are real it will be possible to compare them.

A numerical value of this polynomial up to $n = 5$ is provided in equation (33).

3.3 Numerical value of the polynomial

The polynomial $P(k^2)$ is given below up to $n = 10$. [13]

$$P(k^2) = \frac{\pi}{2}(1 + \frac{3}{16}(k^4) + \frac{105}{1024}(k^8) + \frac{1155}{16384}(k^{12}) + \frac{225225}{4194304}(k^{16}) + \frac{2909907}{67108864}(k^{20})) \tag{33}$$

4 Conclusion

In this paper, we focused our analysis on the space geodesic which does not look to be the most important in the theory of general relativity which is a spacetime theory. But as the most important information that we get in physics about some physical or geometrical objects does not reside in the objects themselves but in their relations, this complementary analysis which provides a set additional relations between objects may induce a better understanding of the underlying physics described by the theory.

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[13] The coefficients B_n are computed by using mathematica line of command : $Bn = FullSimplify[((2n)!/(n!)^2)(2^{-2n}((-1)^n)Sum[((-1/2)^j)Binomial[n,j]Binomial[2j,j],(j,0,n)]]$, for $n = 2, 4, 6, 8, 10$.

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