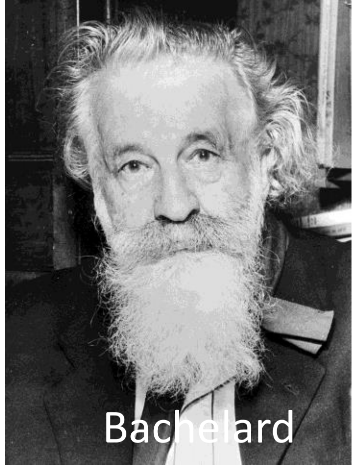
When mathematics reveals hidden physical reality

In 1961 Newman and Penrose proposed a spinor-like formalism for general relativity.
Surprisingly, this formalism will stress the structural role of the light in relativity.

Jacques Fric Paris-Diderot University-Lab. SPHERE; jacques.fric@etu.univ-paris-diderot.fr

Algebra gathers all the information



- In his book, ”The new scientific spirit”, [1] G. Bachelard, after considering the interplaying relations between what he calls « the realism » related to the experience and the « rationalism », related to our mind elaborating theories for making the physical world understandable, declared:
- « As the relations between objects are not substantial but relational, it is the algebra which gathers all relations but only relations. But whether the objects are not the roots of relations, one may wonder from where this relations come»
- This comment was mainly aimed to mathematics, but this also holds for the role of mathematics in some fundamental laws of physics.

Algebra gathers all the information

- So, in mathematics, the compact group of rotations in the 3D Euclidean space acting on vectors is called SO(3). A vector recovers its position after a rotation of 2π radians around an axis in space. But the topology of this group is not simplex. If we write the Lie algebra of the three generators of this group, we discover that there are other solutions.
- A 2D solution, the SU(2) group acting on spinors, where they recover their position after a rotation of 4π instead of 2π, has a simplex topology and is therefore more fundamental than SO(3).
- This shows that the Lie algebra, relating the relations between the objects (the generators of the group), includes all the information about rotation in this space. The SO(3) group allowing to build the Lie algebra was just an example. The Lie algebra extracted its fundamental relational structure.
- It is the group SU(2) which is used in the Electroweak theory which relies on the product O(1).SU(2), proving that SU(2) is more physical than SO(3).

Illustration of the properties of the SU(2) group

These figures come from the book "Gravitation" by C.W Misner, K. S.Thorne and J.A Wheeler [2]

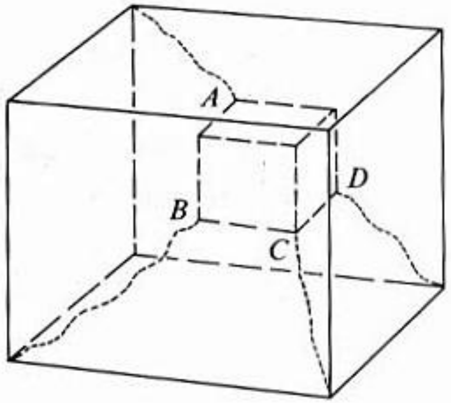


Figure 41.5.
"Orientation-entanglement relation" between a cube and the walls of a room. A 360° rotation of the cube entangles the threads. A 720° rotation might be thought to entangle them still more—but instead makes it possible completely to disentangle them.

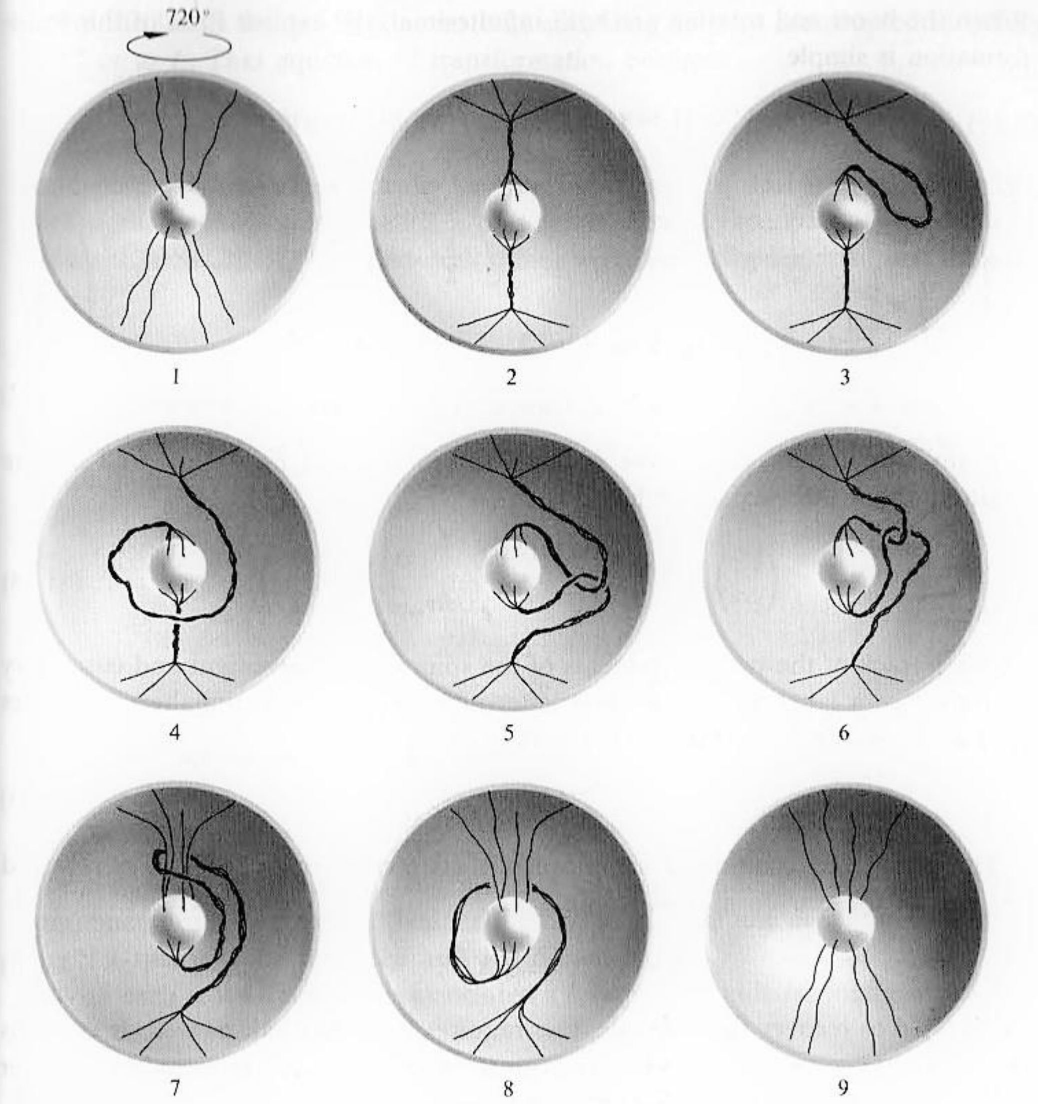
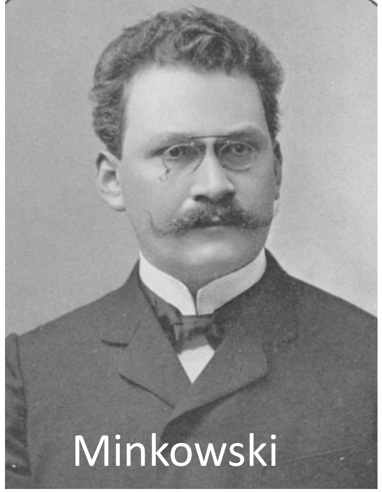


Figure 41.6.
An object is connected to its surroundings by elastic threads as in Figure 41.5. (Eight are shown here; any number could be used.) Rotating the object through 720° and then following the procedure outlined (Edward McDonald) in frames 2–8 (with the object remaining fixed), one finds that the connecting threads are left disentangled, as in frame 9 (lower right).

Mathematics at the rescue of physics



- An other example is provided in relativity by the concept of spacetime.
- After proposing its formalism for special relativity, Minkowski declared that space and time are no longer fundamental concepts and have no physical reality.
- They are only some kind of shadow of a more complex physical reality called spacetime [3].
- Spacetime is something perfectly well defined by the mathematics in special relativity, but it is impossible for a human mind to conceive the concept of spacetime as time and space are so pregant in our mind that the first thing we do when one say spacetime is to try to separate space and time.
- So, at the beginning of special relativity one tried to recover the Newtonian approach by synchonizing clock in a frame for having a universal time in this frame and independantly a space. But this work only in a frame!

Mathematics and physics

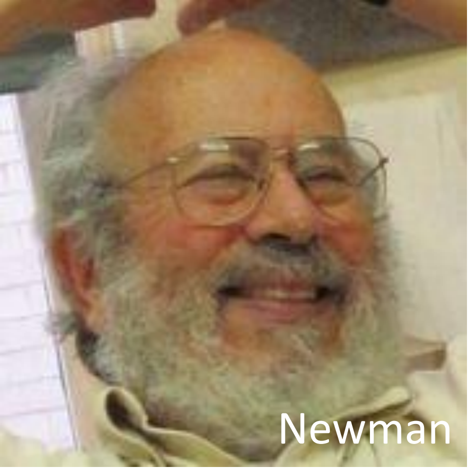
- The role of mathematics in the foundation of physics is a widely debated topic. Many papers have been written about it.
- Here we will work on an example for illustrating how the formalism may be of a great help for explaining the physical phenomenology.
- After the examples that we gave for illustrating G. Bachelard arguments we will describe an other example where the formalism helps to understand the physical phenomenology in relativity.
- We will start by recalling the standard formalism used in relativity in order to stress the originality of the proposal of Newmann and Penrose that we will describe.

Mathematics and physics

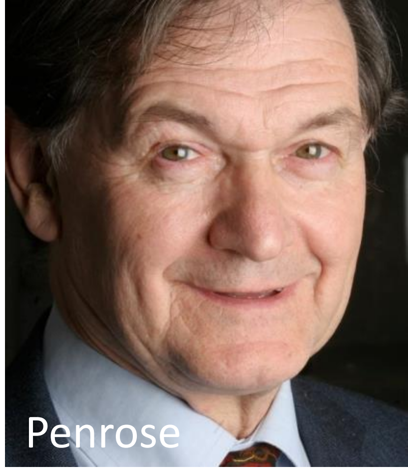
- As the relativity is a geometrical theory of gravitation which describes a spacetime, it is usually described, in analytic geometry by using its tools. On a manifold, modelling the spacetime, a set of global three spacelike and one timelike coordinates, which are functions, allows to define any point.
- As the general relativity is locally compatible with special relativity, it may also be described locally by using a Minkowki local base of orthonormal 4-vectors (3 spacelike and 1 timelike) with local coordinates.
- This, called tetrad formalism, which is sometimes used, alternatively, for describing the spacetime needs a different set of mathematics tools for being achieved.

Introduction to the NP formalism

- A good introduction to the Newmann-Penrose (NP) formalism is given by S. Chandrasekhar in his book « The mathematical theory of black holes » [4]:
- « The Newmann-Penrose formalism is a tetrad formalism with a special choice of the basis vectors. The choice that is made is a tetrad of null vectors l, n, m and m* of which l and n are real and m and m* are complex conjugate of one another.
- The novelty of the formalism, when it was first proposed by Newmann and Penrose in 1962 [5], was precisely in their choice choice of a null basis: it was a departure fromthe choice of an orthonormal basis which was customary till then.



Introduction to the NP formalism



- Penrose was originally led to consider the introduction of a null basis in incorporating in general relativity spinor analysis in an essential way.
- The underlying motivation for the choice of a null basis was Penrose's strong belief that the essential element of a spacetime is its light cone structure which makes possible the introduction of a spinor basis.
- And it will appear that the light cone structure of the spacetime of the black holes solutions of general relativity is exactly of the kind that makes the Newman-Pernrose formalism more effective for grasping the inherent symmetries of these spacetime and revealing their analytic richness. »
- In addition S. Chandrasekhar point out that this formalism is particularly efficient in vacuum for the solutions of general relativity. This includes black holes solutions (more generally solution of Type D in the Petrov-Pironi classification [6,7]) but also special relativity. We will explain why.

Introduction to the NP Formalism

- The word « light » is a generic term for electromagnetic waves or any phenomenon whose velocity is that of light, i.e also gravitational waves.
- More important, it is the fact that, for complying with the relativity principle, there exists a velocity invariant in the relativity formalism, acting as a upper limit for any physical phenomenoly which is the physical structural constraint.
- Therefore, it is not the light itself which rules the causality but the fact that there is an upper limit of velocity, the light being only a kind of messenger traveling at this upper limit of velocity.
- This is this upper limit of velocity which destroys the absolute time and absolute space of Newton mechanics. This makes the relativistic universe totally different from that of Newton as a local perturbation is not immediately known by the whole universe.

Comments on this introduction

- There are several issues in the use the tetrad formalism with a basis made of a set of null vectors instead of 3 spacelike and 1 timelike vector.
- A first issue is that, as each null vector is orthogonal to itself, there are only 3 lineary independant orthogonal null vectors, instead of 4 as in the Minkowski basis. This looks not compatible for describing a problem with 4 degrees of freedom.
- Therefore they defined a set of 2 « real » null vectors l, n and 2 « complex » null vector each one being the conjugate of the other. The real part and the imaginary part will provide 2 additionnal degrees of freedom making the total to 4, as required.
- Let us recall that his requires a different formalism for the derivatives of tensors and vectors (Ricci rotation coefficient instead of Christoffel symbols).

Comments on this introduction

- In relativity the metric is defined by the formula:
  - $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
- In this formula $g_{\mu\nu}$ is callled the metric tensor, dx^μ, dx^ν, the infinitisimal variation of coordinates and *ds* is the line element on the worldline that we will have to integrate by using this formula for getting the « length » (affine parameter) of the worldline..

- A second issue is that, as the ds² is null, in null geodesic ($ds^2 = 0$), we cannot use it as an affine or dynamic parameter on a world line.

Newman-Penrose formalism

- In this new formalism the metric tensor is no longer that of the special relativity.

1			
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- But, instead a spinor-like tensor, see [8] for the demonstration.

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Newman-Penrose formalism

- This is mathematically perfectly defined but it is difficult to give a physical representation of this as a basis of null vectors is something very difficult to conceive.
- How a set of null coordinates, on a basis made of 3 null vectors including one complex null vector, which are paths of the light orthogonal to themselves and that have null affine parameter ($ds^2 = 0$) in a 4 dimensionnal spacetime, can be used for defining points on the manifold?
- We will see that such formalism, which looks very obscure, will in fact simplify the equations in many cases, this meaning that, likely, there is a morphism between the structure (symmetries) of this formalism and the structure of the phenomenology that it describes.

NP formalism simplifies equations

- First example

- In the Kerr [9] or Kerr Newman [10] black holes, in this NP formalism, the the spacetime can be fully defined only by one Weyl complex scalar, instead of several in all other formalisms.
- Let us recall that such spacetime of type D is fully defined in vacuum by the Weyl tensor which is a conformal tensor, outside of the ring singularity. The Weyl tensor, which is the Riemann tensor in vacuum, defines the curvature of the spacetime everywhere. The Weyl tensor has 256 components, but per its symmetries, it has only ten independent values, usually represented by 5 complex scalars called Weyl scalars.
- The fact that only one of them is needed in the NP formalism show that, as declared by S. Chandrasekhar, it is the most efficient for grasping the symmetries of this kind of spacetime.

NP formalism simplifies equations

- Second example
- In special relativity, we may also use the NP formalism. The first step is to write the Minkoski metric (coordinates *t*, *x*, *y*, *z*) :
  - $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$
- in null coordinates *U*, *V*, *W*, *W** by setting:
  - $U = a(t+x)$, $V=a(t-x)$, $W = a(y+iz)$, $W^* = a(y - iz)$
- Where $a = 2^{-1/2}$, is a normalization factor. The metric is now written:
  - $ds^2 = -2dUdV + 2dW.dW^*$
- A boost of parameter φ such that $v/c = \tanh \varphi$ where *v* is the relative velocity and a rotation of parameter θ, are defined by operators which are 4x4 matrix.
- Let us compare these operators in the Minkoski and NP formalisms.

NP formalism simplifies equations

- In Minkoski formalism for a boost along the axis *x* and a rotation around the same axis *x*, the matrix is usually written:

cosh(φ)	-sinh(φ)		
-sinh(φ)	cosh(φ)		
		cos(θ)	sin(θ)
		-sin(θ)	cos(θ)

- In NP formalism for a boost along the axis *x* and a rotation around the same axis *x*, the matrix can be written (see [8] for the demonstration):

$e^{-\varphi}$			
	e^{φ}		
		$e^{-i\theta}$	
			$e^{i\theta}$

- The NP formalism provides a simpler and more symmetrical matrix, grasping again the symmetries of the phenomenology.

Mathematics reveal the NP efficiency

- As the Einstein equation is a set of no linear partial second order differential equations, analytic solutions may be expected only for highly symmetric spacetime. At early time in relativity, the solutions were established by using the symmetries of the spacetime described by the ds^2.
- The Schwarzschild solution in vacuum, in 1916, relies on a generic form of the ds^2 with a spherically space section, constrained by the Einstein equation (the Ricci tensor must vanish) and the convergence with the Newton equation at infinity. This method has allowed to find some solutions of highly symmetric spacetime but failed to find the solution for rotating black holes which was found by Kerr 47 years later!
- Meanwhile, other methods were developed considering the Weyl tensor (which is invariant by a conformal transformation), which fully specifies the curvature of spacetime in vacuum, as an operator acting on bi-vectors. The study of the eigenvalues of the Weyl tensor will provide an other fruitful approach.

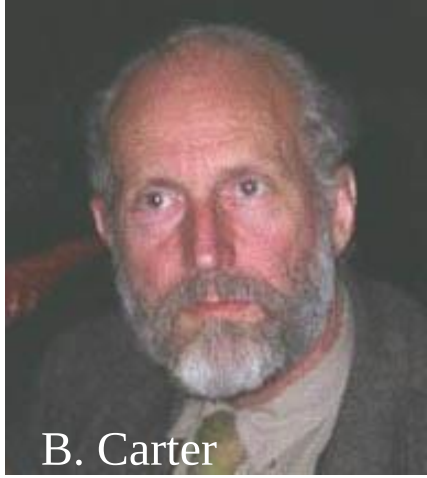
Mathematics reveal the NP efficiency

- An interesting result is that there are a set of 4 null geodesics (bound to the roots of a quadratic equation), called principal null geodesics which play a structural role in the space time as they fully define the metric.
- Categories of vacuum spacetimes will depend on the number of different roots.
- In type D solutions where the metric is defined in vacuum (empty conformal spacetimes), there are 2 double roots, one corresponding to a class (congruence) of null incoming geodesics (going towards the singularity) and the other to a class (congruence) of null outgoing (coming from the singularity) geodesics.

Mathematics reveal the NP efficiency

- If the NP formalism is so efficient for simplifying the equations describing the spacetime, this is not fortuitous, it is because it relies on these classes of principal null geodesics.
- Penrose suspected that the light rays (null geodesics) were playing an essential role as they rule the causality. This was its main motivation in the NP formalism, but there is more information in this formalism. It defined totally the spacetime as the metric tensor can be written as the sum of the Minkowski tensor (a fixed tensor) plus the tensorial product of principal null geodesics.
- This is the way followed by Kerr in his search of a solution, as related by B. Carter in [11]:

Mathematics reveal the NP efficiency



- “ In all these spaces the Weyl tensor is of type D in the Petrov-Pirani classification, the two double principal null vectors being given by ..(equations)…. By the kundt and Trumper generalization of the Goldberg-Sachs theorem, these are integrable to give two shear-free null geodesics congruences. The first of these is incoming the second outgoing…..
- It is by making use of these structural properties of the Weyl tensor and specifically looking for non hypersurface-orthogonal solutions, that the empty space metrics of the family were derived by Kerr. Subsequently these metrics were derived by Kerr and Schild from a systematic study of empty solutions whose metric tensor is ‘locally) the sum of a flat-space metric tensor and the tensor product of a null vector with itself.”

Contribution of E. Cartan in 1922

- In a paper published at the « Académie des Sciences » [12] Elie Cartan, as soon as in 1922, noticed the interest of what he called «the optical universe » which belongs to a class of empty conformal spacetimes in relativity.
- He noticed the existence of a class null geodesics, called principal null geodesics, which play a structural rôle in the description of the Schwarzchild spacetime (which is defined in vacuum).
- He identified that in the Schwarzschild spacetime these four principal null geodesics are only two (each one is double).
- It is because this spacetime is of type D, but Cartan did not know that as this will be established more than 30 years later by Petrov and Pirani in their classification.

This suggest a dual approach in Special relativity

- In Special relativity (SR), as all inertial (Galilean) frames are equivalent (no preferred frame) usually, one is selected as the reference (that of the observer) and for the others, their boost, their rotation are described from its point of view.
- The velocity of light is an invariant (the same for all Galilean frames), but the frequency of a light ray emitted on a frame is different in the others.
- As we know the structural role of light in relativity and as the velocity of light is an invariant, this suggests to select a null geodesic as the reference.

This suggest a dual approach in Special relativity

- As $ds^2 = 0$, usually the four-momentum p^μ is used as affine parameter on null geodesics. It depends on the frequency of the photon whose energy is:
  - $E = h\nu$
- Because:
  - $p^\mu = E/c = h\nu/c$
- In this case, if we calculate the Doppler shift f/f_0 between two frames of relative velocity v by using the relativistic Doppler equation with $v/c = tanh(\varphi)$ i.e $\varphi = argtanh(v/c)$, see [8] for a demonstration, we will get :
  - $\frac{f}{f0} = e^{\varphi}$
- This result which gives the ratio of frequency will give the ratio of affine parameter of two Galilean frames in the frequency representation.

Conclusion

- The Newman-Penrose formalism, in addition to simplify the equations and to reveal hidden symmetries the spacetimes open the door to a new conceptual approach relying on null basis geodesics and null coordinates which is the dual of the usual analysis.
- A frequency-based analysis, is a kind of Fourier formalism where, as many tools have been developed, this open a new field of investigations.

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